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A simple class of $\mathcal{N} = 3$ gauge/gravity duals

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Abstract: We find the gravity duals to an infinite series of $\mathcal{N} = 3$ Chern-Simons quiver theories. They are $\mathrm{AdS}_4 \times M_7$ vacua of M-theory, with M_7 in a certain class of 3-Sasaki-Einstein manifolds obtained by a quotient construction. The field theories can be engineered from a brane configuration; their geometry is summarized by a "hyperKähler toric fan" that can be read off easily from the relative angles of the branes. The singularity at the tip of the cone over M_7 is generically not an orbifold. The simplest new manifolds we consider can be written as the biquotient $\mathrm{U}(1)\backslash\mathrm{U}(3)/\mathrm{U}(1)$. We also comment on the relation between our theories and four-dimensional $\mathcal{N} = 1$ theories with the same quiver.

Keywords: AdS-CFT Correspondence, D-branes, Flux compactifications.

Contents

1. Introduction

The AdS/CFT correspondence is at present best understood for AdS_5. Shortly after the original proposed duality for $\mathrm{AdS}_5 \times S^5$ [1], which is dual to an $\mathcal{N}=4$ CFT_4, some simple $\mathcal{N}=1$ generalizations were put forward [2, 3]. The correspondence was later applied to more general Calabi-Yau cones; today nice combinatorial methods exist [4 – 7] to determine quickly what $\mathcal{N}=1$ theory is dual to a given conical Calabi-Yau.

A similar story for AdS_4 has so far remained more mysterious, although [8 – 10] have predicted certain aspects of the superconformal theory with a crystal model. One reason for this is that, whereas a four-dimensional Yang-Mills coupling $g_{\mathrm{YM}\,4}$ runs logarithmically, with a coefficient that can be tuned to zero with a judicious choice of field contents, its analogue $g_{\mathrm{YM}\,3}$ in three dimensions runs already classically. This is related to the fact that the dilaton for a D2 brane solution is not constant (nor is the near-horizon limit for its supergravity solution of the form $\mathrm{AdS}_4 \times M_6$). Even though the M2 solution does not have these problems, not enough is understood of its superconformal fixed point.

This situation has begun to change recently. Following the discovery of a superconformal $\mathcal{N}=8$ Chern-Simons theory [11 – 14], a version of the correspondence, in the spirit of [15], has been proposed in which the CFT has a Lagrangian description: it is a $\mathcal{N}=6$ superconformal theory [16]. In a certain regime, its gravity dual is $\mathrm{AdS}_4 \times \mathbb{CP}^3$ in IIA. Other solutions dual to CFT's with a Lagrangian descriptions have since been proposed: an orbifold of $\mathrm{AdS}_4 \times \mathbb{CP}^3$ in [17, 18], solutions that come from extrema of the same $\mathcal{N}=8$ four-dimensional effective theory in [18 – 20], cases with orientifolds [21 – 23], and a squashed $\mathbb{CP}^3$ [24]. Earlier proposals for superconformal duals to $\mathcal{N}=3$ AdS_4 have been made in [25 – 28].

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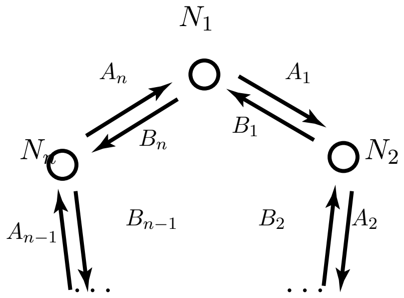

Figure 1: The quiver.

In this note, we propose a series of $\mathcal{N}=3$ quiver theories dual to the near-horizon limit of M2 branes at certain hyperKähler singularities. Even if the engineering of our theories is a straightforward variation on [16], we believe it is important to enlarge the class of duals available for study. We regard our results as a first step towards obtaining as rich a variety of $\mathrm{AdS}_4/\mathrm{CFT}_3$ duals as presently available for $\mathrm{AdS}_5/\mathrm{CFT}_4$.

We describe the moduli spaces of these quiver theories as hyperKähler quotients, which allows us to show in an elegant way their equivalence to the proposed dual backgrounds.

In section 2 we will introduce our $\mathcal{N}=3$ superconformal quiver theories. In 2.2 we compute their moduli space; while this has been done before [29], we will point out that these manifolds are hyperKähler toric, so that their geometry is summarized neatly by an array of two-dimensional vectors (see figure 2 below). We also consider the orientifold of these theories.

In section 3 we then engineer the theories in string theory (again as done in [29]); it turns out that this can be done with a configuration of branes whose inclinations are the same as the vectors in the hypertoric fan. We use a duality chain [30] to explain that coincidence, and hence to derive infinitely many instances of the AdS/CFT correspondence. We also discuss the reduction to IIA theory on $\mathrm{AdS}_4 \times M_6$ in the large K limit.

2. The theories and their moduli space

We will consider $\mathcal{N}=3$ Chern-Simons actions. Such theories are rigid, in the sense that the Lagrangian is fixed by the gauge groups, Chern-Simons levels, and matter representations. Moreover, such an $\mathcal{N}=3$ theory can be constructed given such data [31]. The gauge group is $\mathrm{U}(N_1)\times\ldots\times\mathrm{U}(N_n)$; there is also matter, in the form of $\mathcal{N}=2$ multiplets A_i, B_i ("chiral", in the sense that they depend only on $\theta\equiv\theta^1+i\theta^2$), for $i=1,\ldots,n$.

These fields transform in the bifundamentals of the gauge group, in the way shown in figure 1.

The requirement that there be $\mathcal{N}=3$ supersymmetry fixes the action:

$$S=\sum_i\left[S_{\mathcal{N}=2\,\mathrm{CS}}(k_i,V_i)+\int d^4\theta\,\mathrm{Tr}\Big(e^{-V_i}A_i^\dagger e^{V_{i+1}}A_i+e^{V_i}B_i e^{-V_{i+1}}B_i^\dagger\Big)+\int d^2\theta W\right]\;; \quad (2.1)$$

the $\mathcal{N}=2$ superpotential W is

$$W=\sum_{i=1}^{n}\frac{1}{k_i}Tr(B_iA_i-A_{i-1}B_{i-1})^2\ . \tag{2.2}$$

The levels k_i, $i=1,\dots,n$ satisfy

$$\sum_{i=1}^{n}k_i=0\ . \tag{2.3}$$

The $\mathcal{N}=2$ Chern-Simons action was introduced in [32, 33], and shown to be conformal in [34, 31]; the general structure of the $\mathcal{N}=3$ Chern-Simons theories was first developed in [35–37].

We will engineer these actions from string theories in section 3 (as also done in [29]). For the time being, however, we will study their moduli spaces and show that they suggest a gauge-gravity duality.

2.1 The moduli space

We will first look at the moduli space in the case in which all the gauge group ranks are one: $N_i=1$, $i=1,\dots,n$.

By varying (2.2), one gets the F-terms[1]

$$(k_i+k_{i+1})A_iB_i=k_iA_{i+1}B_{i+1}+k_{i+1}A_{i-1}B_{i-1}\ . \tag{2.4}$$

The D-term equation have a similar form, with A_iB_i replaced by $|A_i|^2-|B_i|^2$. One can then write D-term and F-term together as

$$(k_i+k_{i+1})\,q_i^\dagger\sigma_\alpha q_i=k_i\,q_{i+1}^\dagger\sigma_\alpha q_{i+1}+k_{i+1}\,q_{i-1}^\dagger\sigma_\alpha q_{i-1}\ , \tag{2.5}$$

where $q_i=(A_i,\bar{B}_i)^t$ and σ_α are the Pauli matrices.[2] It is also convenient to write this as

$$0=\sum_j M_{ij}\,q_j^\dagger\sigma_\alpha q_j\ ,\qquad M=\begin{pmatrix} k_1+k_2 & -k_1 & 0 & \cdots & 0 & -k_2\\ -k_3 & k_2+k_3 & -k_2 & 0 & \cdots & 0\\ 0 & -k_4 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & -k_n & k_{n-1}+k_n & -k_{n-1}\\ -k_n & 0 & \cdots & 0 & -k_1 & k_1+k_n \end{pmatrix}\ . \tag{2.6}$$

If one remembers that $\sum_i k_i=0$, it is easy to see that this matrix has rank $n-2$. Hence, at this point, we have found $3(n-2)$ independent real equations on $4n$ real coordinates.

We now turn to the gauge fields. One might think that the moduli space obtained so far has to be quotiented by $U(1)^n$. However, it is easy to see that the vector $\mathcal{A}_+\equiv\sum_i\mathcal{A}_i$

[1] When one varies (2.2) with respect to A_i, one obtains (2.4) multiplied by B_i, and vice versa; this might raise the question of whether we are overlooking other branches, but one can show a posteriori that this is not the case.

[2] One can use quaternions already at the level of the action if one uses $\mathcal{N}=1$ superfields; in the abelian case, the potential reads $\int d^2\theta_1\sum_i\frac{1}{k_i}(q_i^\dagger\sigma_\alpha q_i-q_{i-1}^\dagger\sigma_\alpha q_{i-1})^2$.

does not couple to any scalar. It then turns out that one of the remaining $n-1$ gauge transformations gets discretized by the Chern-Simons coupling $\sum_i k_i \mathcal{A}_i F_i$.

Let us see this more in detail. The gauge transformations act as

$$\mathcal{A}_i \to \mathcal{A}_i + d\lambda_i \; ; \qquad A_i \to e^{i(\lambda_i - \lambda_{i+1})} A_i \, , \qquad B_i \to e^{i(\lambda_{i+1} - \lambda_i)} B_i \; . \tag{2.7}$$

It can be helpful to think of these transformations by going to a new basis of vectors:

$$\mathcal{A}_+ = \sum_{i=1}^{n} \mathcal{A}_i \, , \qquad \tilde{\mathcal{A}}_i = k_{i+1} \mathcal{A}_i - k_i \mathcal{A}_{i+1} \, , \qquad \mathcal{A}_- = \sum_{i=1}^{n-1} \mathcal{A}_i - \mathcal{A}_n \; . \tag{2.8}$$

$\mathcal{A}_+$ does not act on the scalars; as for for the gauge transformation of $\tilde{\mathcal{A}}_i$, the i-th row of M gives the charges under it of the j-th scalar. Only $n-2$ of the $\tilde{\mathcal{A}}_i$ are independent (for the same reason that M had rank $n-2$), and so we complete the basis with $\mathcal{A}_-$. Both $\mathcal{A}_\pm$ do not couple to themselves but to each other: in this basis, there is a term $k_n \mathcal{A}_+ d\mathcal{A}_-$, but no $\mathcal{A}_+ d\mathcal{A}_+$ nor $\mathcal{A}_- d\mathcal{A}_-$. Since the choice of $\mathcal{A}_-$ in (2.8) might seem arbitrary, let us describe the situation a bit more invariantly. We can think of $\int \sum_i k_i \mathcal{A}_i \wedge d\mathcal{A}_i$ as a quadratic form $\mathcal{K} = \mathrm{diag}(k_1, \ldots, k_n)$ pairing the vector $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ to itself. Then $\mathcal{A}_\pm$ are orthogonal to themselves because they are in the kernel of $\mathcal{K}$ (thanks to $\sum_i k_i = 0$); and the $\tilde{\mathcal{A}}_i$ in (2.8) are orthogonal to $\mathcal{A}_\pm$ for $i = 2, \ldots, n-1$.

In any case, the constant gauge transformations generated by $\mathcal{A}_-$ do not quotient the moduli space. We can see this in the same way as in [38, 16], with $\mathcal{A}_\pm$ playing the role of A_b, $A_{\bar{b}}$ in that reference. Since $\mathcal{A}_+$ only appears through the coupling $k_n \mathcal{A}_+ d\mathcal{A}_-$, we can dualize it to a periodic scalar τ, and add it to the configuration space. This scalar is charged under $\mathcal{A}_-$, and it can be used to fix its gauge transformation (up to a discrete component that we will analyze shortly). This means that only the gauge transformations of the $\tilde{\mathcal{A}}_i$ should act on the moduli space. Hence the matrix of charges can be taken to be M; since this matrix has rank $n-2$, if one wants to avoid redundancies, one can erase from it the first and last rows.

Let us now look at the possible residual gauge symmetry λ_- from $\mathcal{A}_-$. Looking at (2.7), we see that it acts with charge 1 on A_{n-1}, with charge -1 on A_n, and 0 on the remaining A_i. τ has period 2π, but transforms as $\tau \to \tau + k_n \lambda_-$ (just as in [16]). This results in a discrete action on A_{n-1} and A_n. One might, however, worry that this result depends on the particular choice of $\mathcal{A}_-$ we have made in (2.8). We can actually summarize both the discrete part of the gauge transformations and the $\mathrm{U}(1)^{n-2}$ by defining the group

$$\mathbf{N} \equiv \mathrm{Ker}(\beta) \, , \tag{2.9}$$

where

$$\beta : \mathrm{U}(1)^n \to \mathrm{U}(1)^2 \qquad \beta = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ k_1 & k_1 + k_2 & \ldots & k_1 + \ldots + k_{n-1} & 0 \end{pmatrix} \; . \tag{2.10}$$

Note that the ambiguity in defining β by taking different linear combinations of its rows results in a map with the same kernel. Moreover, when we later relate the column of this matrix to the fivebrane charges used to engineer the field theory, this ambiguity is precisely the choice of an $\mathrm{Sl}(2, \mathbb{Z})$ duality frame.

Let us check that $\mathbf{N}$ is the right residual gauge group. An element $(e^{i\lambda_1}, \ldots, e^{i\lambda_n}) \in \mathbf{N}$ has to satisfy

$$e^{i\sum_i p_i\lambda_i} = 1\,, \qquad e^{i\sum_i \lambda_i} = 1\,, \tag{2.11}$$

where

$$p_1 = k_1\,, \qquad p_2 = k_1 + k_2\,, \ldots \qquad p_{n-1} = k_1 + \ldots + k_{n-1}, \qquad p_n = 0 \tag{2.12}$$

are the entries on the first row of β, (2.10). There is a continuous component to its general solution: we can take $\theta_i = \sum_{j=2}^{n-1} M_{ji}\hat{\theta}_j$, since, as remarked above, the rows of M generate the kernel (over $\mathbb{R}$) of β. The $\hat{\theta}$ span a subgroup $\mathrm{U}(1)^{n-2} \subset \mathbf{N}$, which corresponds to the gauge transformation for the vectors $-k_{j-1}\mathcal{A}_j + k_j\mathcal{A}_{j+1}$.

Let us now look at the discrete part of $\mathbf{N}$. For example, in the case considered in [16], β is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & k \end{pmatrix}\,. \tag{2.13}$$

In this case, (2.11) gives $e^{ki\lambda_2} = 1$, $e^{i(\lambda_1+\lambda_2)} = 1$, which is solved by $e^{i\lambda_1} = e^{-i\lambda_2} = \omega_k^j$, where ω_k is a k-th root of unity and $j = 1, \ldots, k$. In this case $\mathbf{N} = \mathbb{Z}_k$ (there is clearly no continuous component) and the moduli space is $\mathbb{C}^2/\mathbb{Z}_k$.

In the general case, one can consider for example the action of the gauge transformation corresponding to the vector $\mathcal{A}_-$ above. This acts only on the last two hypermultiplets: $\lambda_i = 0$ but for λ_{n-1} and λ_n. This is the residual discrete gauge invariance of $\mathcal{A}_-$ mentioned earlier. Equation (2.11) now can be solved by $e^{i\lambda_{n-1}} = e^{-i\lambda_n} = \omega_{k_n}^j$ (with ω_{k_n} a k_n-th root of unity, and $j = 1, \ldots, k_n$). This would seem a discrete subgroup of $\mathbf{N}$. However, when the k_i are coprime, one can see that this discrete subgroup is actually a subgroup of the $\mathrm{U}(1)^{n-2}$ that we already considered. When the k_i are not coprime, there is a genuinely discrete component to $\mathbf{N}$, that we can take to be generated by $\mathbf{n} = (0, \ldots, 0, \omega_K, \omega_K^{-1})$, where K is the l.c.d. of the k_i, and ω_K is once again a K-th root of unity. We will actually see in section 3 that the case in which the k_i are not coprime has a particular relevance to us.

We have seen that, for $n > 2$, $\mathbf{N}$ will necessarily have a continuous part. One might wonder, though, whether one could get rid of some of the coordinates and keep only the discrete part, so that the singularity in the moduli space is still an orbifold. We will see in the next subsection that this is typically not the case.

Summing up, the moduli space that we have obtained in this section is given by

$$\mathcal{M} = \frac{\{\; q_i = (A_i, \bar{B}_i) \in \mathbb{C}^{2\cdot n} \;|\; \sum_j M_{ij} q_j^\dagger \sigma_\alpha q_j = 0 \;\}}{(A_i, B_i) \sim (\mathbf{n}_i A_i, \mathbf{n}_i^{-1} B_i)\,,\ \ \mathbf{n} \in \mathbf{N}} \tag{2.14}$$

where M has been defined in (2.6) and $\mathbf{N}$ has been defined in (2.9), (2.10). Notice that the numerator consists, as mentioned earlier, of $3(n-2)$ real equations; and that the denominator is a group whose continuous part is $(n-2)$-dimensional. Hence one expects a dimension

$$4n - 3(n-2) - (n-2) = 8\ . \tag{2.15}$$

In fact, we will now see that this quotient has already been studied by mathematicians, and that one can say a great deal more about $\mathcal{M}$.

2.2 Geometrical interpretation

Let us first take a step back. An easy way to produce a Kähler manifold of complex dimension $k - j$ is the so-called Kähler quotient construction. One starts from $\mathbb{C}^k$, with $\mathrm{U}(1)^j$ acting on it. Using the Kähler form as a symplectic form, one can compute a Hamiltonian μ for each of the j $\mathrm{U}(1)$ actions; this j-uple of functions is called the moment map for the $\mathrm{U}(1)^j$ action. Since the latter preserves the Hamiltonian, it acts in particular on each of the level sets $\mu = \mu_0$. So if one considers

$$\mathbb{C}^k//(\mathbb{C}^*)^j \equiv \frac{\{\mu = \mu_0\}}{\mathrm{U}(1)^j}\,, \tag{2.16}$$

one ends up with a manifold of complex dimension $k - j$. One can then "pull back" the Kähler structure from $\mathbb{C}^k$ to the quotient (2.16), which is then a Kähler manifold itself.

There is a similar procedure for hyperKähler manifolds. One starts now from $\mathbb{H}^k$, again with a $\mathrm{U}(1)^j$ action. One now has an SU(2) triplet of moment maps $\vec{\mu}$ (a "hyperKähler moment map" [39–41, 30]). One can then quotient by $\mathrm{U}(1)^j$:

$$\frac{\{\vec{\mu} = \vec{\mu_0}\}}{\mathrm{U}(1)^j}\,, \tag{2.17}$$

similarly to (2.16). This time, one has lost $3j + j$ real coordinates, or j quaternionic coordinates. It turns out that the manifold defined in this way can be given a hyperKähler structure.

Let us see the quotient (2.17) more in detail, following in particular [41]. First of all, let us think of our $q_i = (A_i, \bar{B}_i)$ as the i-th quaternionic coordinate on $\mathbb{H}^n$. We can act with a natural U(1) on each of these coordinates: the i-th such action reads

$$(A_i, \bar{B}_i) \to (e^{i\theta} A_i, e^{i\theta} \bar{B}_i)\ . \tag{2.18}$$

The hyperKähler moment map that generates an action is the triplet

$$q_i^\dagger \sigma_\alpha q_i\ . \tag{2.19}$$

If one has many U(1) actions acting on the q_i according to a charge matrix M, one ends up with a hyperKähler moment map $M_{ij} q_i^\dagger \sigma_\alpha q_i$, which is exactly what enters in (2.14).

One can think of each of these moment maps as a different Hamiltonian for the U(1) actions. It is then not a surprise that the U(1) actions leave the value of $M_{ij} q_i^\dagger \sigma_\alpha q_i$ invariant. In particular, the locus in the numerator of (2.14) is left invariant by the action in the denominator. Hence we see that (2.14) is in fact a particular case of the general definition of hyperKähler quotient (2.17) [41].

We have actually glossed over the fact that our action is not exactly an $\mathrm{U}(1)^{n-2}$ action, but it might have a discrete component: the abelian group in the denominator of (2.14) was defined in (2.9). Fortunately, the definition in [41] considers exactly the same type of abelian action, and hence we can use their results.

To say more, let us introduce some more notation. Call $u_i \in \mathbb{R}^2$ the columns of (2.10). (These vectors are the analogue of the fan in toric geometry.) In terms of the p_i defined in (2.12), they look like figure 2:

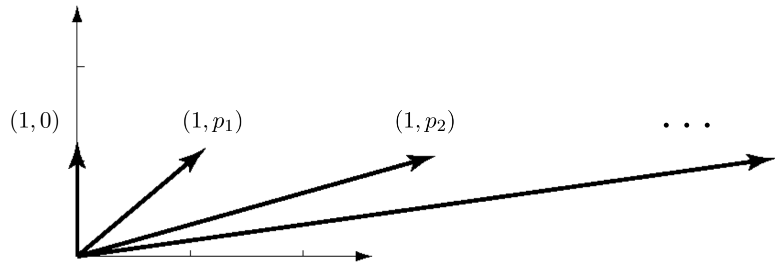

Figure 2: The vectors u_i of the "hypertoric fan".

We have remarked already that in our case $\mathcal{M}$ is a cone. At the tip of this cone one will find a singularity, but one would like the base of the cone to be nonsingular. Theorem 4.1 in [41] guarantees that this is the case for our β, as long as $k_i \neq 0$ for all $i = 1, \ldots, n$. As for the conical singularity, Theorem 3.3 in [41] gives a criterion to decide when it is an abelian orbifold: in our case, the criterion says that the singularity is an abelian orbifold if and only if there are only two different u_i (the columns of β). This is in agreement with what we know about the $\mathcal{N} = 6$ theory in [16] (for which $n = 2$, and hence there are indeed only two u_i, namely the two columns of (2.13)), and for the theory in [18] (in which $k_{\rm odd} = k$ and $k_{\rm even} = -k$, so that $u_{\rm odd} = \binom{1}{k}$ and $u_{\rm even} = \binom{1}{0}$).

The "base" (or horizon) B_7 of the cone,

$$\mathcal{M} \equiv \mathrm{Cone}(B_7)\,, \tag{2.20}$$

is called 3-Sasakian whenever the cone $\mathcal{M}$ is hyperKähler (which is our case). The B_7 we are getting in this paper were defined and studied by Boyer, Galicki and Mann [42, 43], as part of a more general construction; a review of the part relevant to us is given in [44, section 7].

2.3 Comparison with four-dimensional quiver theory

In this paper we are using quivers to define $\mathcal{N} = 3$ theories in three dimensions. Quivers have been used for a long time to define $\mathcal{N} = 1$ theories in four dimensions, and it is natural to wonder if there is any relation between the moduli spaces in the two cases. In this subsection, we will call the moduli space defined in (2.14) (the one relevant for this paper) $\mathcal{M}_{\rm d=3}$. Likewise, we will call $\mathcal{M}_{\rm d=4}$ the moduli space for the $d = 4$, $\mathcal{N} = 1$ theory defined from the same quiver and superpotential.

At first blush, one might think that there should be little relation between the two. For example, for the $d = 3$, $\mathcal{N} = 6$ theory in [16], $\mathcal{M}_{\rm d=3} = \mathbb{C}^4/\mathbb{Z}_k$; the $d = 4$, $\mathcal{N} = 1$ theory with the same quiver and superpotential has $\mathcal{M}_{\rm d=4} =$ the conifold [3]. These two spaces appear to be very different.

The difference between $\mathcal{M}_{\rm d=3}$ and $\mathcal{M}_{\rm d=4}$ can be traced to the D-term contribution to the potential. In four dimensions, the D-term equation for our quiver would have read

$$|A_i|^2 - |B_i|^2 - |A_{i-1}|^2 + |B_{i-1}|^2 = \mu_i \qquad \text{(in four dimensions)} \tag{2.21}$$

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with μ_i a Fayet-Iliopoulos parameter. In our case, the D-term component ($\alpha = 3$) of (2.6) can be written as

$$\frac{|A_i|^2 - |B_i|^2 - |A_{i-1}|^2 + |B_{i-1}|^2}{k_i} = \frac{|A_{i+1}|^2 - |B_{i+1}|^2 - |A_i|^2 + |B_i|^2}{k_{i+1}} \,. \tag{2.22}$$

We can rewrite this as

$$|A_i|^2 - |B_i|^2 - |A_{i-1}|^2 + |B_{i-1}|^2 = \lambda\, k_i\,, \qquad \forall i\,, \tag{2.23}$$

for some λ. (2.23) is now formally identical to (2.21). The difference is that λ is an arbitrary parameter, whereas the μ_i in (2.21) have to be fixed to some value (and are moduli for the geometry).

Another, related, difference between the two theories arises in the action by the gauge group. In the four-dimensional theory, one mods out by $\mathrm{U}(1)^{n-1}$; in the three-dimensional theory, by $\mathrm{U}(1)^{n-2}$.

Putting these two remarks together, we can say that

$$\mathcal{M}_{\mathrm{d}=4} = \mathcal{M}_{\mathrm{d}=3}//\mathbb{C}^* \tag{2.24}$$

or, in other words, that $\mathcal{M}_{\mathrm{d}=3}/\mathrm{U}(1)$ is foliated by copies of $\mathcal{M}_{\mathrm{d}=4}$ obtained for different values of the four-dimensional FI parameters $\mu_i = \lambda k_i$. (The leaves will change topology at some special value of the λ.)

This relation to the moduli spaces of the $3+1$ Yang-Mills theory with the same quiver persists when the Chern-Simons-matter theory only possesses $\mathcal{N} = 2$ supersymmetry. In that case, the D-terms are the same as in (2.23), but the F-term equations are different.

It is also interesting to consider turning on (generically $\mathcal{N} = 2$) FI parameters in the Chern-Simons theory. The quaternionic equations 2.5 are modified to

$$\frac{1}{k_{i+1}}(q_i^\dagger \sigma_\alpha q_i - q_{i+1}^\dagger \sigma_\alpha q_{i+1}) - \zeta_{\alpha,\,i+1} = \frac{1}{k_i}(q_{i-1}^\dagger \sigma_\alpha q_{i-1} \; - q_i^\dagger \sigma_\alpha q_i) - \zeta_{\alpha,\,i} \,. \tag{2.25}$$

The $\alpha = 3$ component of (2.25) is again implied by (2.21) with $\mu_i = (\zeta_{3,\,i} + \lambda)k_i$, so again the moduli space of the 2+1 theory is a Kähler quotient of the 3+1 moduli space. (Nonzero $\zeta_{1\,i}$ and $\zeta_{2\,i}$ would come from a modification of the $\mathcal{N} = 2$ superpotential (2.2).)

For example, in the case of the quiver with two nodes, used in three dimensions in [16], and in four dimensions by [3], the statement (2.24) is, for $k = 1$,

$$\mathbb{C}^4//\mathbb{C}^* = \text{conifold} \;: \tag{2.26}$$

if one takes $\mathbb{C}^4/\mathrm{U}(1)$ and one fixes λ, one obtains a copy of the resolved conifold for $\lambda \neq 0$, and of the singular conifold for $\lambda = 0$; by varying λ, one sweeps the entire $\mathbb{C}^4/\mathrm{U}(1)$.

The quivers in figure 1 were used for four-dimensional $\mathcal{N} = 1$ theories in [45]. Their description of the theory is in terms of two "big matrices" X_1, X_2 whose blocks summarize our A_i and B_i respectively, and of a third, block-diagonal, matrix Φ. Their superpotential is

$$W = \mathrm{Tr}\Big(\Phi[X_1, X_2] + M\Phi^2\Big)\,, \tag{2.27}$$

where M is also block-diagonal; if one integrates out Φ, one obtains our superpotential (2.2), with their m_i equal to our k_i. The moduli spaces for [45] are

$$\mathcal{M}_{\rm d=4} = \{ \ (u, v, z, w) \in \mathbb{C}^4 \mid uv = \Pi_{i=1}^n (z - k_i w) \ \} \, , \tag{2.28}$$

which they call "generalized conifolds". We can then give a description of our $\mathcal{M}_{\rm d=3}$ defined in (2.6), by reading (2.24) backwards.

2.4 An example

We will now use [44] to gain insight on perhaps the simplest new set of examples discussed in this paper: the case in which the number of nodes $n = 3$. As we pointed out, we know from [41, Theorem 3.3] that in this case the conical singularity is not an abelian orbifold. In this case, M has only one row (repeated three times):

$$(k_3 \ k_1 \ k_2) \tag{2.29}$$

which is the "charge vector": its i-th entry tells us the charge of $(A_i, \bar{B}_i)$ under the only U(1). There is then only one triplet of equations, and only one quotient:[3]

$$\mathcal{M}_{n=3} = \frac{\{ \ (A_i, B_i) \mid \ \sum_i k_{i-1} A_i B_i = 0 \, , \sum_i k_{i-1}(|A_i|^2 - |B_i|^2) = 0 \ \}}{(A_i, B_i) \to (e^{ik_{i-1}\theta} A_i, e^{-ik_{i-1}\theta} B_i)} \ . \tag{2.30}$$

This manifold is a cone. One would be tempted to fix the radial gauge by imposing (say) the further constraint $\sum_i k_{i-1}(|A_i|^2 + |B_i|^2) = 2$. Since $\sum_i k_i = 0$, the k_i cannot all be positive, and one would end up with hyperboloids that are not particularly illuminating. One can, however, change variables and obtain clearer equations. Suppose for example $k_{1,2} > 0$, $k_3 < 0$. If one defines

$$k'_3 = -k_3 \, , \qquad k'_{1,2} = k_{1,2} \ ; \qquad A'_3 = -B_3 \, , \qquad B'_3 = A_3 \, , \tag{2.31}$$

equation (2.30) is invariant in form:

$$\mathcal{M}_{n=3} = \frac{\{ \ (A'_i, B'_i) \mid \ \sum_i k'_{i-1} A'_i B'_i = 0 \, , \sum_i k'_{i-1}(|A'_i|^2 - |B'_i|^2) = 0 \ \}}{(A'_i, B'_i) \to (e^{ik'_{i-1}\theta} A'_i, e^{-ik'_{i-1}\theta} B'_i)} \ ; \tag{2.32}$$

with the difference that now

$$k'_i > 0 \ . \tag{2.33}$$

This change of variables is equivalent to flipping the sign of one vector in the hypertoric fan. Such an operator does not alter the resulting hyperKähler geometry, which can be understood in terms on the (p, q) 5-branes we shall soon discuss, as the fact that the angle at which an anti-fivebrane must lie to preserve $\mathcal{N} = 3$ supersymmetry is exactly the opposite of the corresponding fivebrane; hence they are the same object.

[3] For simplicity we assume here that the k_i are coprime. The case in which they are not is a discrete quotient of the case in which they are, as in (3.8).

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We can now study the base of the cone by intersecting it with the further constraint $\sum_i k'_{i-1}(|A'_i|^2 + |B'_i|^2) = 2$. One obtains

$$\sum_i k'_{i-1}|A'_i|^2 = \sum_i k'_{i-1}|B'_i|^2 = 1\,, \qquad \sum_i k'_{i-1}A'_i B'_i = 0\ . \tag{2.34}$$

If we define $x_i = \sqrt{k'_{i-1}}A'_i$, $y_i = \sqrt{k'_{i-1}}\bar{B}'_i$, equation (2.34) is equivalent to the statement that x and y can be used as the first two columns of a U(3) matrix U. There is a U(1) worth of vectors $\in \mathbb{C}^3$ that can be used a third column of U. Hence, solutions to (2.34) are given by the quotient U(3)/U(1). Remembering now the action on the quotient of (2.32), and reinstating the conical direction, we conclude that

$$\mathcal{M}_{n=3} = \mathrm{Cone}(B_7)\,, \qquad B_7 \equiv \mathrm{U}(1)\backslash\mathrm{U}(3)/\mathrm{U}(1)\ . \tag{2.35}$$

The left action is given by $\mathrm{diag}(e^{ik'_3\theta}, e^{ik'_1\theta}, e^{ik'_2\theta})$, and the right action is given by $\mathrm{diag}(1, 1, e^{i\phi})$. Notice that, since each of the U(1) has determinant not equal to one, one cannot rewrite (2.35) as SU(3)/U(1) nor as U(1)\SU(3).

The base B_7 of the cone $\mathcal{M}_{n=3}$ in (2.35) is sometimes called "Eschenburg space" [46]; it was later found in [44] (where it was called $\mathcal{S}(k'_3, k'_2, k'_1)$) that it is indeed 3-Sasakian — or in other words, that the cone $\mathcal{M}_{n=3}$ over B_7 is indeed hyperKähler. B_7 in (2.35) is a close relative to the Aloff-Wallach spaces $N(k, l)$ considered in the early Kaluza-Klein literature [47–49]; those spaces are $\mathcal{N} = 3$ only for $k = l = 1$ (and $\mathcal{N} = 1$ otherwise), whereas (2.35) is $\mathcal{N} = 3$ for any k'_i. Only $N(1, 1)$ is a particular case of Eschenburg space, but it would require $k_1 = k_2 = k'_3 = 1$, which does not occur for us.

Various aspects of the topology and geometry of (2.35) have been studied in [46, 50, 44, 51, 27], where in particular it is shown that its non-zero cohomology groups are $H^2(B_7, \mathbb{Z}) = H^5(B_7, \mathbb{Z}) = \mathbb{Z}$, $H^4(B_7, \mathbb{Z}) = \mathbb{Z}_{k_1k_2+k_1k'_3+k_2k'_3}$. [27, 28] also considered the problem of finding their field theory dual.

The isometry group of (2.35) is U(2) × U(1). In the field theory, an SU(2) subgroup of this appears as an R-symmetry; a U(1) is the symmetry generated by the current $*\mathcal{F}_+$ (which has now become a total symmetry); a further U(1) is due to the symmetry $(A_i, B_i) \to (e^{i\theta}A_i, e^{-i\theta}B_i)$.

2.5 More general quivers?

The techniques introduced (or reviewed) so far in this section can be actually applied to more general $\mathcal{N} = 3$ theories. Even though the string theory interpretation is as yet problematic, as we will explain later, the moduli spaces can be computed just as easily as for the quiver in figure 1.

Let us consider a more general quiver (with bifundamental matter). In this subsection, the index a will run over the nodes and the index i over the hypermultiplets. The scalars in each hypermultiplet q_i again make up a complex doublet $(A_i, \bar{B}_i)$. We encode the quiver in a charge matrix X_{ai} that has value 1 if A_i leaves the node a, −1 if it exits from it, and 0 otherwise. The F- and D-terms then read

$$M^{\mathrm{gen}}_{ij} q^\dagger_j \sigma_\alpha q_j = 0\,, \qquad M^{\mathrm{gen}} \equiv \sum_a \frac{1}{k_a} X_{ai}X_{aj}. \tag{2.36}$$

The U(1) actions turn out to involve the same matrix, and one obtains once again a hyperKähler quotient. Let us work it out.

The matrix M^{gen} in (2.36) has a zero eigenvalue for any (independent) loop in the quiver. For generic k_a, then, the rank of the matrix is #(edges) − #(loops). Since the number of variables is #(edges), it follows that the dimension is

$$\text{dimension} = 4\,\#(\text{loops})\ ; \tag{2.37}$$

hence, for string theory applications, one would need a quiver with two loops. The hypertoric fan is easy to work out. Each of the two rows can be computed from one of the two loops. If one chooses an overall orientation for the loop, the i-th entry of the row is 1 if the field A_i follows the orientation, −1 if it runs opposite to it, and 0 if the edge does not belong to the loop. The $\mathcal{N} = 3$ theory considered in [25] is of this type; it has $N(1,1) = \mathrm{SU}(3)/\mathrm{U}(1)$ as an abelian moduli space. ([26] also considered a similar theory, although only in the abelian case.) At the end of subsection 2.4 we commented on how $N(1,1)$ is different from (2.35).

There is another possibility to consider. If one also imposes

$$\sum_a k_a = 0\,, \tag{2.38}$$

the rank of the matrix M^{gen} in (2.36) drops further by 1, and one obtains that

$$\text{dimension} = 4\Big(\#(\text{loops}) + 1\Big)\ . \tag{2.39}$$

The theories (2.1) are of this type. The matrix M^{gen} is in that case essentially M in (2.6) (except that, in (2.6), we have multiplied the i-th row by $k_i + k_{i+1}$, for better readability). The fan can again be easily worked out. One of the rows corresponds to the single loop in the quiver. The other row can be obtained by considering a path connecting all nodes, and it depends on the levels k_a in a way similar to the one in (2.10). To summarize, the abelian moduli space of a general quiver $\mathcal{N} = 3$ Chern-Simons is still a hyperKähler quotient. The reason we are only sketching these general rules is that the use of these quivers in string theory is not yet entirely clear. There are clear candidates to engineer these theories, suggested by the hypertoric fan along the lines described in the next section. However, checking that the engineering of the theories actually works would involve understanding the general rules for extracting an effective theory for D3's intersecting general (p,q)-fivebranes, which are not at the moment entirely clear.

2.6 Orientifold theories and their moduli space

In this section we will consider orientifolding the theories introduced above. Later this will be related to a brane construction with an O3 plane wrapped around the circle with the D3 branes. We begin with an $\mathcal{N} = 3$ quiver Chern-Simons theory of the type defined by (2.1), such that the number of nodes is even, and the levels are given by $2k_i$. The reason for these restrictions will be seen shortly. This generalizes the orientifolds considered in [22, 23]. Here

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there is no enhanced supersymmetry beyond $\mathcal{N}=3$ to begin with, hence the orientifolded theory preserves the same supersymmetries.

The action of the orientifold on the hypermultiplets, $C_i=(A_i,\ B_i^\dagger)$, is given by

$$\begin{aligned} C_{A\ 2i+1} &\to -M_{AB}\, C^*_{B\ 2i+1}\, J\,, \\ C_{A\ 2i} &\to -M_{AB}\, J\, C^*_{B\ 2i}\,, \end{aligned} \tag{2.40}$$

where $M_{AB}=i\sigma_2$, and J is the invariant anti-symmetric matrix of the USp theory. It is easy to check that this is a discrete symmetry (for even ranks) of the action (2.1), which we now gauge.

The gauge groups become, alternatively, $\mathrm{O}(2N_{2i+1})$ and $\mathrm{USp}(2N_{2i})$, and the projected matter fields are naturally thought of as hypermultiplets obeying a reality condition. As shown in [23] the levels become $2k_{2i+1}$ and k_{2i}, which is why the original levels were chosen to be even. To be more explicit, we can take the identification on the matter fields to be $B_{2i+1}=J\,A^T_{2i+1}$ and $B_{2i}=A^T_{2i}\,J$. Therefore the $\mathcal{N}=2$ superpotential is given by

$$W=\sum_{i=1}^{n}\left[\frac{1}{2k_{2i-1}}\mathrm{Tr}(JA^T_{2i-1}A_{2i-1}-A_{2i-2}A^T_{2i-2}J)^2+\frac{1}{2k_{2i}}\mathrm{Tr}(A^T_{2i}JA_{2i}-A_{2i-1}JA^T_{2i-1})^2\right]. \tag{2.41}$$

The moduli space of a single whole M2 brane in this background corresponds to taking all of the ranks equal to two, so that the gauge groups are $\mathrm{O}(2)=\mathrm{U}(1)\rtimes\mathbb{Z}_2$ and $\mathrm{USp}(2)=\mathrm{SU}(2)$. Generalizing the analysis of [23], we will choose $A_i=x_iI+y_iJ$. Now all of the matrices appearing in the F-term and D-term equations commute, since $A_i^T=x_i-y_iJ$. To relate the resulting constraints to the hyperKähler quotient described above, it is convenient to change variables to $u_i=x_i+iy_i$ and $v_i=x_i-iy_i$, so that the combination that appears in the F-terms, $A_i^TA_i=Ju_iv_i$, and in the D-terms, $A^\dagger_{2i-1}A_{2i-1}-A^T_{2i-1}A^*_{2i-1}=J(|u_{2i-1}|^2-|v_{2i-1}|^2)$, $A^\dagger_{2i}A_{2i}+JA^T_{2i}A^*_{2i}J=J(|u_{2i}|^2-|v_{2i}|^2)$.

Therefore, defining $q_i=(u_i,\ v_i^*)$, we see that the moduli space is described by equations identical to 2.5. The gauge symmetry is broken to $\mathrm{U}(1)^{2n}$, given by the $\mathrm{SO}(2)$ and the $\mathrm{U}(1)$ subgroup of $\mathrm{USp}(2)$ generated by $i\sigma_2$. Therefore we can apply the same reasoning as before to obtain a hyperKähler quotient.

There is an additional discrete quotient obtained as in [23] from the $\mathbb{Z}_2$ component of the $\mathrm{O}(2)$ gauge groups, given by the element σ_1. In general, acting by this transformation will not preserve the ansatz above, but a simultaneous action in all $O(2)$ factors, together with a particular $USp(2)$ transformation does act on the moduli space. Its action is $A_{2j}\to i\sigma_3A_{2j}\sigma_1$ and $A_{2j+1}\to\sigma_1A_{2j+1}i\sigma_3$ hence $q_i=(u_i,\ v_i)\to(-v_i,\ u_i)$. Note that applying this transformation twice is just $q_i\to-q_i$, which is already quotiented, since the least common divisor of the levels is at least two, and this transformation is present in the discrete component of the hyperKähler quotient.

3. String theory interpretation

We have defined in (2.1) a certain set of $\mathcal{N}=3$ theories, and we have shown that their moduli space $\mathcal{M}$ (defined in (2.14)) is a (toric) hyperKähler manifold of dimension 8. This is a cone over a 3-Sasakian manifold B_7.

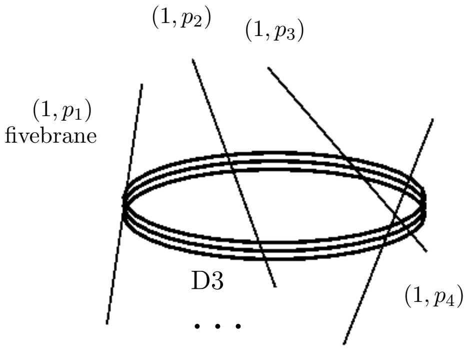

Figure 3: A cartoon of the brane configuration.

This fact suggests that these theories might actually be dual to $\mathrm{AdS}_4 \times B_7$. We will now show that this is indeed the case. The argument is very similar to the one in [16].

We start with a brane configuration in IIB. As shown in figure 3, it consists of a D3 brane along directions 0126 (where x^6 is actually compactified to a circle), and of several fivebranes. The i-th of these fivebranes has charges $(1, p_i)$, and is extended along $012[37]_{\theta_i}[48]_{\theta_i}[59]_{\theta_i}$, where $\tan(\theta_i) = p_i$. This configuration preserves $\mathcal{N} = 3$ supersymmetry [52, 53].

Using the same reasoning as in [16], one can see that this configuration is such that it engineers the theory (2.1), after relating the p_i to the k_i as in (2.12). In fact, we can remark already now that the u_i of the "hypertoric fan" in figure 2 have coordinates exactly equal to the (p, q) charges of the fivebranes in figure 3.[4] This coincidence will be explained shortly.

If one takes the figure literally, so that the D3 all fill up the entire x^6 circle, the configuration actually engineers only the case in which all the ranks are equal, $N_i = N_{\mathrm{D3}}$, $i = 1, \ldots, n$. One could allow for D3 suspended between the fivebranes, but for simplicity we will not do that, and in what follows we will consider $N_i = N$.

One can now T-dualize this configuration along direction 6 and then lift it to M-theory, again similarly to [16]. For the time being, we forget about the D3 branes; we will reintroduce them later. The details of the duality chain are explained in [30]. Jumping at the result, the metric that one gets in M-theory is of the form

$$ds^2 = ds^2_{\mathrm{Mink}_3} + U_{ab} d\mathbf{x}^a \cdot d\mathbf{x}^b + U^{ab}(d\phi_a + A_a)(d\phi_b + A_b)\,, \tag{3.1}$$

where $a, b = 1, 2$. The internal eight-dimensional metric is a T^2 fibration over $\mathbb{R}^6$ that generalizes to higher dimension the Taub-NUT metric. The equations on the two-by-two matrix U and on A for a general metric of the form (3.1) to be hyperKähler are also similar to the Taub-NUT case. The particular solution one gets in our case is given by [41, 30]

$$U_{ab} = U^{\infty}_{ab} + \sum_{i=1}^{n} \frac{u_{ia} u_{ib}}{|\vec{x} + p_i \vec{y}|} \tag{3.2}$$

[4]This is reminiscent of the way some of the theories in [26] were engineered.

where $\vec{x}, \vec{y}$ are coordinates on $\mathbb{R}^6$ (namely, directions $3, 4, 5$ and $7, 8, 9$ respectively); recall that for us the $u_i = \binom{1}{p_i}$ are the columns of β. Notice also that

$$\vec{x} + p_i \vec{y} = q_i^\dagger \vec{\sigma} q_i \tag{3.3}$$

is the general solution to (2.6), as becomes apparent by looking at (2.10) and (2.12), whose rows generate the kernel of M.

One can think of the T^2 in (3.1) as follows. We started our discussion of hyperKähler quotients in section 2.2 by considering the n U(1) actions (2.18). Then we quotiented by a subgroup $\mathbf{N}$ (defined in (2.9) of that $\mathrm{U}(1)^n$. Since the matrix M has for us rank $n-2$, $\mathbf{N}$ has a continuous part $\mathrm{U}(1)^{n-2}$, which leaves us with an unquotiented $\mathrm{U}(1)^2 = T^2$. This is the T^2 in (3.1). In fact, by looking at the degeneration loci

$$|\vec{x} + p_i \vec{y}| = 0 \,, \quad i = 1, \ldots, n \,, \tag{3.4}$$

we can conclude that $\phi^2 = x^{10}$ (the M-theory circle), and that $\phi^1 = \tilde{x}^6$, the T-dual of the original x^6 direction in the IIB configuration (the compact direction in figure 3).

Now, near each of the loci (3.4), the metric (3.2) in M-theory is non-singular: it behaves like a single Taub-NUT solution. One expects, however, a singularity at the origin, where all those loci coincide. If we focus on this singularity, we can drop the term U_∞ from (3.2):

$$U_\infty = 0 \,. \tag{3.5}$$

We can now recognize that the metric found in [41, Theorem 9.1] as being the one induced on our $\mathcal{M}$ by the hyperKähler quotient procedure is the same as (3.1). Therefore we have shown that the moduli space of the conformal field theory is exactly the geometry obtained by dualizing and lifting the fivebranes to M-theory.

The last step is to reinstate the D3 branes. When we do include them, we end up with a stack of M2 branes at the tip of $\mathcal{M}$. In the near-horizon limit for these M2 branes, one ends up with

$$\mathrm{AdS}_4 \times B_7 \tag{3.6}$$

(recall that $\mathrm{Cone}(B_7) = \mathcal{M}$), with Freund-Rubin fluxes. Since B_7 is a 3-Sasakian manifold, this solution is $\mathcal{N} = 3$.

Let us summarize the results so far. The configuration of branes in figure 3 engineers the theories (2.1) with the quiver in figure 1. A chain of duality from that brane configuration gives us a space that, with the help of [41, Theorem 9.1], we recognize as (3.6), with B_7 such that $\mathcal{M} = \mathrm{Cone}(B_7)$ is exactly the same space that we had obtained as a moduli space of the same theories $\mathcal{M}$.

In other words, we have derived a gauge/gravity duality between the theories (2.1) and the solutions (3.6), with $\mathcal{M} = \mathrm{Cone}(B_7)$ defined in (2.6).

A similar construction also engineers the theories in section 2.6. Together with the D3 branes stretched along the x^6 direction as in figure 3, one now introduces an O3 projection inverting the direction $3, 4, 5, 7, 8, 9$ (namely, $\vec{x}$ and $\vec{y}$ in (3.2)). There actually exist many types of O3 projections (usually called $\mathrm{O3}^\pm$, $\widetilde{\mathrm{O3}}^\pm$), and we have to specify the one we mean.

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In fact, the type of orientifold plane changes as one crosses a brane [54, 55]. For example, let us consider the case in which all the p_i in figure 3 are even. Then it is consistent to have the O3 to be of type $O3^-$ between the i-th and $(i+1)$-th fivebrane, and of type $O3^+$ between the $(i-1)$-th and i-th fivebrane. This configuration engineers the theories considered in section 2.6, with superpotential (2.41). One can then follow a similar duality chain as the one we considered in the case without orientifold. One ends up with a space of the type we just considered (namely, (3.6), with $\mathcal{M}$ defined in (2.6)), but further quotiented by the action $\vec{x} \to -\vec{x}$, $\vec{y} \to -\vec{y}$.

Let us now go back to the case without orientifolds. Just like in [16], we can take a limit in which the gauge group ranks $N_i = N$ are large, and the 't Hooft couplings $\lambda_i = N/k_i$ are large; in that limit, the appropriate description is in type IIA. One way to take the k_i large is to rescale them all simultaneously:

$$k_i = K\tilde{k}_i \tag{3.7}$$

with the $\tilde{k}_i$ coprime. We already remarked in section 2.1 that the group $\mathbf{N}$ (defined in (2.9)) has then a discrete component. In fact, it turns out that

$$\mathcal{M}(K) = (\mathcal{M}(K=1))/\mathbb{Z}_K \tag{3.8}$$

where $\mathbb{Z}_K$ acts on ϕ^1 (recall that the metric on $\mathcal{M}$ is given in (3.1).

We can now take $K \to \infty$. Since the size of ϕ^1 shrinks to smaller and smaller size, the solution is best described in IIA:

$$\mathrm{AdS}_4 \times M_6\,, \qquad M_6 = B_7/\mathrm{U}(1)\ . \tag{3.9}$$

Let us give more details about M_6. One might worry that the U(1) quotient in (3.9) might introduce singularities in the IIA solution.

In fact, this does not happen for the particular U(1) we are quotienting. Let us see how an orbifold singularity could arise. If, at a certain locus, the U(1) orbit happened to close for $\phi^1 = \frac{2\pi}{k}$, for k some integer, then there would be a $\mathbb{Z}_k$ singularity in the quotient. For a standard example of such a phenomenon, consider the sphere S^{2n-1} described by the locus $\sum_{i=1}^n a_i|z_i|^2 = 1$ in $\mathbb{C}^n$, and quotient it by the action $z_i \to e^{ia_i\theta}z_i$. At the point $z_2 = \ldots = z_n = 0$, for example, the U(1) action closes at $\theta = \frac{2\pi}{a_1}$. The quotient is indeed the weighted projective space $\mathbb{W}P(a_1, \ldots, a_n)$, which has an orbifold singularity at the point $z_2 = \ldots = z_n = 0$.

Now, for us, the loci to worry about are the ones defined in (3.4). At each of those, the cycle $(1, p_i)$ of the T^2 degenerates. The U(1) action we are considering is along the cycle $(0, 1)$. Since $(0, 1)$ and $(1, p_i)$ make up together a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, the U(1) action does not close before 2π, and there is no orbifold singularity. Notice that this would not have been the case if we had considered the diagonal action, along cycle $(1, 1)$. This action is in many ways more natural from a mathematical point of view, and in particular it is part of the triple of "Reeb vectors" that can be used to define a 3-Sasaki structure [44, Prop. 1.2.2]. Indeed, it is almost never the case that one of those vectors yields a quotient without orbifold singularities (see [44, Theorem 4.4.5, Prop. 1.2.10]). Fortunately, we managed to avoid that possible problem.

Physically, we actually remarked earlier that the branes only coincide at the origin; this is another way of seeing that there should be no singularity in M_6.

Even though we have not studied in detail the topology of M_6, looking at the brane configuration one would expect $n-1$ 2-cycles. The period of F_2 on the i-th 2-cycle should then be p_i.

We can say more about M_6 again in the $n=3$ case. The relevant $\mathcal{M}$ and B_7 were described in section 2.4; when reducing to IIA, we obtain

$$M_{6\,,n=3} = (\mathrm{U}(1)\times \mathrm{U}(1))\backslash \mathrm{U}(3)/\mathrm{U}(1)\ . \tag{3.10}$$

Its isometry group is $\mathrm{SU}(2)\times\mathrm{U}(1)$, and the cohomogeneity is 2. Together with the fact that we are not reducing on a Reeb vector of the 3-Sasakian structure, this indicates that there is no reason in this case for the dilaton to be constant.

In conclusion, we have seen that the theories defined in (2.1) are dual to a IIA configuration of the form (3.9), with $\mathcal{M} = \mathrm{Cone}(B_7)$ defined in (2.6).

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